Generalized Weierstrass kernels on the intersection of two complex hypersurfaces

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Abstract

On plane algebraic curves the so-called Weierstrass kernel plays the same role of the Cauchy kernel on the complex plane. A straightforward prescription to construct the Weierstrass kernel has been known for more than one century. How can it be extended to the case of more general curves obtained from the intersection of hypersurfaces in an n-dimensional complex space? This problem is solved in this work in the case $n = 3$. As an application, the correlation functions of bosonic string theories are constructed on a canonical curve of genus four. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Algebraic curves and n-sheeted branched covers on the projective line $\mathbf{P}^1$ provide an explicit representation of abstract Riemann surfaces. Besides being an active field of research in mathematics [1–9], for more than a decade, they have been also successfully applied in several different topics of theoretical and mathematical physics. [2]

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[2] Some applications of: Hyperelliptic curves in perturbative string theory [10–22]; algebraic curves in various non-perturbative aspects of string theory and related problems [23–32]; algebraic curves in conformal field theories [33–41]; algebraic curves in polymer physics [42–44]. Other applications of algebraic curves [45–52].

All the above physical applications involve only plane affine algebraic curves. Moreover, most of the works are further restricted to the particular case of hyperelliptic curves. The reason is that the latter are very well known in mathematical literature. An invaluable source of hyperelliptic formulas can be found for instance in the original Weierstrass lecture notes [53]. As an example, we will give in Section 3 the prescription of Weierstrass for differentials of the second kind. Already in the case of the so-called Z_n symmetric curves, which are the most straightforward generalization of hyperelliptic curves, superstring calculations like those of [10–22] are impossible, because it has not yet been understood how to construct sections with half-integer spins or chiral determinants [54–56] of free fermions with given spin structures. An exception is provided by more exotic fermions with $1/n$ characteristics, which have been solved in [57]. Also neglecting fermions and sticking to bosonic string theory, the computation of the partition function remains a very complicated task on non-hyperelliptic algebraic curves.

The explicitness of algebraic curves is indeed of great advantage in understanding the physical aspects of theories like superstrings whenever Riemann surfaces are involved, but, on the other side, it is also the main responsible of the above-mentioned difficulties. Using different approaches, like for instance theta functions, the problems are somewhat hidden, but they reappear in other forms, in particular when one needs to explore a limited portion of the moduli space of Riemann surfaces such as that spanned by Z_n symmetric curves.

A considerable effort to understand conformal field theories on plane algebraic curves and to find new applications of non-hyperelliptic curves has been made by the author over the past ten years, partly in collaboration with Sobczyk and Urbanik. In particular, the applications of branched covers of $\mathbf{P}^1$ have been investigated, in which the curves are projected on the complex projective line. Every compact Riemann surface can be represented in this way. Branched coverings are also closely related to affine curves in the two-dimensional complex plane $\mathbf{C}^2$. The bosonic string theories on a general non-hyperelliptic algebraic curve of genus three have been treated in Refs. [58,59], computing the correlation functions of the theory and its partition function, the latter up to a theta constant. Moreover, the partition function has been exactly derived on Z_3 symmetric curves in [59]. Even in this case, where the chiral determinants are well known [60], the calculation of the partition function is not simple. Indeed, one has still to use the Rauch's variational method [61] combined with the Beilinson–Manin formula [62] in order to determine the explicit form of the period matrix. The same variational approach of Rauch has been exploited in [63,64] to rederive in an elegant way the Thomae formulae of Bershadsky and Radul [60] on Z_n symmetric curves.

Despite the difficulties of evaluating the partition functions of conformal field theories, algebraic curves become very convenient in the construction of meromorphic tensors with poles and zeros of given order at given points, like for example the correlation functions of the free fields appearing in the bosonic string action [54]. On general plane algebraic curves, this problem has been solved for the b–c systems in [65,66]. The more complicated case of scalar fields has been treated recently in [67]. The two and four-point amplitudes of the bosonic β–γ systems have instead been derived in [68]. Other useful formulas can be found in [69–71].

Plane algebraic curves are described by complex coordinates which, depending on the interpretation, can be multivalued functions on the complex plane $\mathbf{C}$ or on the complex sphere

$\mathbf{P}^1$. They change their branches at the branch points according to certain monodromy properties. For this reason, it was thought in a first moment that meromorphic tensors on an algebraic curve could be easily expanded in terms of the solutions of the related Riemann monodromy problem (see, e.g. [72]). However, this approach has encountered outstanding obstacles in its concrete realization. Up to now, the best way to expand meromorphic tensors and to handle their singularity structure is provided by the generalized Laurent series of [65,66]. To explain their usefulness with an example, one should recall that the main building block in the construction of the correlation functions of b–c systems and scalar fields is the Weierstrass kernel [53]. The latter is a differential of the third kind which, on the curve, plays the same role of the Cauchy kernel on the complex plane. It is characterized by a simple pole in an assigned point and can be explicitly derived with an algorithm due to Weierstrass. However, besides the desired pole, there are also several spurious singularities, which should be eliminated by subtracting suitable counterterms [3]. A general procedure for this subtraction, which in principle strongly depends on the form of the curve, has been derived expanding the Weierstrass kernel in generalized Laurent series. As well, an operator formalism on plane algebraic curves has been established in [65,66].

If the plane curve is regarded as a Riemann surface, the generalized Laurent series can be considered as a multipoint generalization of the Krichever–Novikov bases [73,74] much in the spirit of Dick et al. [75,76]. On the other side, the modes entering in the expansion of meromorphic tensors consist in a finite set of tensors which are multivalued in $\mathbf{C}$ (or $\mathbf{P}^1$). The latter have a deeper significance than modes defined on abstract Riemann surfaces, since they may be interpreted as correlators of exotic conformal field theories on the complex plane (or sphere). Theories of this kind should be both invariant under the Virasoro group of conformal transformations and under the monodromy group $\mathcal{M}$ of the original curve. The amplitudes of the class of exotic conformal field theories corresponding to $\mathcal{M} = Z_n$ have been constructed in [77] in terms of free bosonized b–c systems. Due to the above mentioned difficulties in computing partition functions, instead, only the N-point functions with $N > 2$ can be derived if $\mathcal{M}$ is non-Abelian. The case $\mathcal{M} = D_n$ has been explicitly worked out in [78]. In this way, a concrete realization of the more general program on holonomic quantum field theories developed by Sato et al. [79–83] has been achieved. The connections between generalized Laurent series and solutions of the Riemann monodromy problem have been partly explored in [78]. Some glimpses of non-commutativity in string theory have been anticipated in [84].

Despite of the above progress in understanding conformal field theories, plane algebraic curves suffer of some limitations. To obtain for instance general non-hyperelliptic Riemann surfaces of genus $g > 3$, they are not powerful enough and it is necessary to consider algebraic curves in the complex projective space $\mathbf{P}^{g-1}$. Moreover, the smallest dimensional space in which a curve may be smoothly embedded is $\mathbf{P}^3$. In attempting to treat b–c systems and scalar fields on non-plane curves, an immediate difficulty arises, because the analog of the Weierstrass kernel is not known. As explained before, this kernel is a crucial ingredient in the derivation of the N-point correlations functions and of any other meromorphic tensor. This problem is solved here in the particular case of algebraic curves in $\mathbf{P}^3$. A simple expression of the generalized Weierstrass kernel on these curves is obtained and, as an application, the correlation functions of the b–c systems and of the scalar fields are computed

on a canonical curve of genus four. Let us notice that curves in $\mathbf{P}^3$ include the interesting subset containing the branched covers of Riemann surfaces.

Since it is too difficult to work on a projective variety, the generalization of the Weierstrass kernel has been derived in local form after restricting oneself to local open sets of $\mathbf{P}^3$, where it is possible to use Euclidean coordinates. This is equivalent to consider affine algebraic curves in $\mathbf{C}^3$. Of course, affine algebraic curves are non-compact, but it is still possible to relate them to compact Riemann surfaces by taking into account also the point at infinity and hence considering the extended complex plane $\mathbf{C} \cup \infty \equiv \mathbf{P}^1$. A similar strategy has been followed in the construction of the standard Weierstrass kernel, defined in [53] on affine algebraic curves in $\mathbf{C}^2$ or on a branched cover of $\mathbf{P}^1$.

The material presented in this paper is divided as follows. Section 2 consists in a brief and elementary introduction to the concept of algebraic curves. In Section 3, the derivation of the standard Weierstrass kernel on plane curves is reviewed and its main properties are discussed. A formula of Weierstrass to build a differential of the second kind on hyperelliptic curves is also presented. It is useful to recall this formula because it seems to have no track in the modern literature, but it is very important whenever current–current interactions between scalar fields have to be taken into account (see, e.g. [33–41,71]). Section 4 contains the derivation of the generalized Weierstrass kernel and a discussion of its main properties. In general, we have been able to show that spurious poles can only appear at points in which the coordinates describing the algebraic curve in $\mathbf{C}^3$ become infinitely large. In Section 5, we treat the particular case of a canonical curve of genus four in detail. As an application, the correlation functions of the b–c systems and of the scalar fields are computed at genus four in Section 6. These theories have been already studied using several different approaches, so they provide a good testing ground for the generalized Weierstrass kernel of Section 4. Surface integrals over algebraic curves are expressed as integrals in $\mathbf{C}^3$ in the presence of Dirac δ-functions in Appendix A. The final comments and conclusions are presented in Section 7.

2. Riemann surfaces as algebraic varieties

Let $\mathbf{P}^n$ denote the complex projective space parameterized by coordinates $\xi = \xi_0, \ldots, \xi_n$. A homogeneous polynomial $F(\xi)$ of degree d defines an algebraic hypersurface on $\mathbf{P}^n$ as the locus of the points $\xi \in \mathbf{P}^n$ satisfying the relation

$$F(\xi) = 0. \tag{2.1}$$

The hypersurfaces are called quadrics if $d = 2$, cubics if $d = 3$, quartics if $d = 4$, etc. A *projective algebraic variety* $\mathcal{V}$ is a subset of $\mathbf{P}^n$ characterized as the intersection of many hypersurfaces

$$\mathcal{V} = \{\xi \in \mathbf{P}^n | F_1(\xi) = \cdots = F_k(\xi) = 0\}, \tag{2.2}$$

where $F_1, \ldots, F_k, k \leq n$, represent a set of homogeneous polynomials of degree $d_1, \ldots, d_k$, respectively. If $k = n - 1$, $\mathcal{V}$ describes a *projective algebraic curve C* in $\mathbf{P}^n$. A point $p \in \mathcal{V}$ is *smooth* [85] if the Jacobian matrix

$$\mathcal{J} = \left[\frac{\partial(F_1, \ldots, F_k)}{\partial(\xi_0, \ldots, \xi_n)} \right] \tag{2.3}$$

has

$$\text{rank}[\mathcal{J}] = k. \tag{2.4}$$

Closely related to projective algebraic varieties are the *affine algebraic varieties*:

$$\mathcal{V}_0 = \{x \in \mathbf{C}^n | f_1(x) = \cdots = f_k(x) = 0\}, \tag{2.5}$$

$f_1, \ldots, f_k$ are polynomials with complex coefficients and $x = x_1, \ldots, x_n$ denotes a set of variables in the n-dimensional complex space $\mathbf{C}^n$. The definition of smooth points on affine varieties is analogous to that of smooth points on projective varieties.

It is often convenient to study projective varieties on local open patches of $\mathbf{P}^n$, where it is possible to use Euclidean coordinates. For instance, for $\sigma = 0, \ldots, n$, one may identify $\mathbf{C}^n$ with the following open subsets of $\mathbf{P}^n$ [85]:

$$U_\sigma = \{[\xi_0, \ldots, \xi_n] \in \mathbf{P}^n | \xi_\sigma \neq 0\} \tag{2.6}$$

via the homeomorphism $\phi : U_\sigma \to \mathbf{C}^n$:

$$\phi[\xi_0, \ldots, \xi_n] = \left(\frac{\xi_0}{\xi_\sigma}, \ldots, \frac{\xi_{\sigma-1}}{\xi_\sigma}, 1, \frac{\xi_{\sigma+1}}{\xi_\sigma}, \ldots, \frac{\xi_n}{\xi_\sigma} \right) \tag{2.7}$$

with inverse

$$(x_1, \ldots, x_n) \to [x_1, \ldots, x_\sigma, 1, x_{\sigma+1}, \ldots, x_n]. \tag{2.8}$$

To simplify the notations, we will consider hereafter only the case $\sigma = 0$. In doing that there is no loss of generality, since none of the variables $\xi_1, \ldots, \xi_n$ plays a privileged role in the present context. Clearly, upon the identifications

$$x_1 = \frac{\xi_1}{\xi_0}, \ldots, x_n = \frac{\xi_n}{\xi_0}, \tag{2.9}$$

$$f_a(x_1, \ldots, x_n) \equiv F_a(1, x_1, \ldots, x_n), \quad a = 1, \ldots, k, \tag{2.10}$$

the restriction of a projective variety $\mathcal{V}$ on U_0 is equivalent to an affine variety on $\mathbf{C}^n$ associated to the system of equations

$$f_a(x_1, \ldots, x_n) = 0, \quad a = 1, \ldots, k. \tag{2.11}$$

In the special case $k = n - 1$, a projective curve and the affine curve related to it via the homeomorphism (2.7) differ only by a finite number of points "at infinity". The latter are determined by the conditions $F_1 = \cdots = F_{n-1} = \xi_0 = 0$, in which ϕ is no longer defined. To show that, we consider a projective algebraic curve

$$C = \{\xi \in \mathbf{P}^n | F_1(\xi) = \cdots = F_{n-1}(\xi) = 0\}. \tag{2.12}$$

Since the F_a are homogeneous polynomials of degree d_a, $a = 1, \ldots, n - 1$, they can be written as follows:

$$F_a(\xi) = \sum_{i_1, \ldots, i_n = 0}^{d_a} (\xi_1)^{i_1} \cdots (\xi_n)^{i_n} (\xi_0)^{d_a - (i_n + \cdots + i_1)}. \tag{2.13}$$

If $\xi_0 = 0$, the system of algebraic equations defining C is trivial unless the condition $d_a - (i_n + \cdots + i_1) = 0$ is fulfilled. Thus, the points on the curve corresponding to $\xi_0 = 0$ are given by the residual system of equations

$$\sum_{i_1,\ldots,i_{n-1}=0}^{d_a} (\xi_1)^{i_1} \cdots (\xi_{n-1})^{i_{n-1}} (\xi_n)^{d_a-(i_{n-1}+\cdots+i_1)} = 0 \tag{2.14}$$

for $a = 1, \ldots, n - 1$. The above relations describe the intersection of $n - 1$ hypersurfaces in $\mathbf{P}^{n-1}$, which, by Bézout theorem, contains a finite number of $d_1, d_2, \ldots, d_{n-1}$ common points as desired.

Algebraic curves are particularly important in the study of compact *Riemann surfaces*. The simplest representation of a Riemann surface Σ_g of genus g is in terms of *plane projective algebraic curves*, or hypersurfaces in $\mathbf{P}^2$ associated to a single algebraic equation of the kind

$$F(\xi_0, \xi_1, \xi_2) = 0. \tag{2.15}$$

A plane curve is said *non-singular* or *regular* provided the condition below is never verified

$$F = \frac{\partial F}{\partial \xi_0} = \frac{\partial F}{\partial \xi_1} = \frac{\partial F}{\partial \xi_2} = 0. \tag{2.16}$$

Modulo conformal transformations, any compact Riemann surface coincides with a plane projective algebraic curve.

A plane curve C can be projected from a point $p \notin C$ onto a complex line L in $\mathbf{P}^2$, which does not contain p. After a linear change of coordinates one may take $p = [0, 0, 1]$ and $L = \{\xi \in \mathbf{P}^2 | \xi_2 = 0\}$. The result of the projection is a representation of the Riemann surface as a *branched cover* (or *n-sheeted covering*) of $\mathbf{P}^1$, whose points are given by the zeros of the complex polynomial $f(x_1, x_2) = F(1, x_1, x_2)$ in the Euclidean coordinates $x_1 = \xi_1/\xi_0$ and $x_2 = \xi_2/\xi_0$. Solving the equation

$$f(x_1, x_2) = 0 \tag{2.17}$$

with respect to x_2, one obtains a multivalued function $x_2(x_1)$ of $x_1 \in \mathbf{P}^1$. The finite *branch points* of $x_2(x_1)$ satisfy the relation

$$f(x_1, x_2) = f^{(0,1)}(x_1, x_2) = 0, \tag{2.18}$$

where we have used the notation

$$f^{(m,n)}(x_1, x_2) \equiv \frac{\partial^m}{\partial x_1^m} \frac{\partial^n}{\partial x_2^n} f(x_1, x_2). \tag{2.19}$$

Starting from a multivalued function, a Riemann surface Σ_g can be constructed in terms of sheets and cross-cuts. In analogy with Eq. (2.16), a cover of $\mathbf{P}^1$ is non-singular if the following identities are never satisfied simultaneously:

$$f = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0. \tag{2.20}$$

The double-sheeted coverings of $\mathbf{P}^1$, or *hyperelliptic curves*, have very special properties with respect to the other curves. Their polynomial equation can always be reduced to the following one:

$$x_2^2 = P_{2g+2}(x_1), \tag{2.21}$$

where $P_{2g+2}(x_1)$ is a polynomial of degree $2g + 2$ in x_1 with complex coefficients. Exploiting the group $SL(2, \mathbf{C})$ of automorphisms of the sphere $\mathbf{P}^1$, the number of independent coefficients reduces to $2g - 1$. Since a general Riemann surface of genus $g > 2$ depends on $3g - 3$ complex parameters, called the *moduli*, not all Riemann surfaces can be hyperelliptic. As a matter of fact, hyperelliptic curves form only a subset of dimension $2g - 1$ in the moduli space [1]. On the other side, plane algebraic curves of genus $g \geq 3$ may be also non-hyperelliptic, but the number of independent parameters which is possible to accommodate in the defining polynomial $F(\xi)$ of Eq. (2.15) is less than $3g - 3$ when $g > 3$. For this reason, in order to construct general non-hyperelliptic Riemann surfaces, one usually considers *canonical curves* embedded in $\mathbf{P}^{g-1}$.

A canonical map φ from a Riemann surface Σ_g to the projective space $\mathbf{P}^{g-1}$ can be established with the help of a basis of holomorphic differentials $\omega_1, \ldots, \omega_g$ on Σ_g as follows:

$$\varphi : p \in \Sigma_g \to [\omega_1, \ldots, \omega_g] \in \mathbf{P}^{g-1}. \tag{2.22}$$

If Σ_g is a general non-hyperelliptic Riemann surface, $C = \varphi(\Sigma_g)$ is called a canonical curve. For instance, a canonical curve of genus $g = 3$ is a plane projective algebraic curve of degree four. The canonical curve with $g = 4$ is instead given by the complete intersection of a quadric and a cubic in $\mathbf{P}^3$. For $g = 5$, the canonical curve is a complete intersection of three quadrics in $\mathbf{P}^4$, etc. A more detailed classifications of canonical curves together with a thorough discussion of some exceptional cases can be found in [4].

Here, we will limit ourselves to algebraic curves in $\mathbf{P}^3$, which is the smallest dimensional projective space in which a curve can be smoothly embedded. In fact, smooth embeddings are not possible in $\mathbf{P}^2$, so that plane algebraic curves are affected by singularities at isolated points. [3] Let $F(\xi)$ and $G(\xi)$, $\xi = \xi_0, \xi_1, \xi_2, \xi_3$ be two homogeneous polynomials of degrees d_F and d_G, respectively, whose zeros define two hypersurfaces in $\mathbf{P}^3$. We are mainly interested in situations in which the intersection of the two hypersurfaces is *complete*, i.e., they meet in a single curve C, whose points are given by the following system of algebraic equations:

$$F(\xi_0, \xi_1, \xi_2, \xi_3) = G(\xi_0, \xi_1, \xi_2, \xi_3) = 0. \tag{2.23}$$

We also assume that all the points of C are smooth. The genus g of C is then given by

$$2g - 2 = d_F d_G (d_F + d_G - 4). \tag{2.24}$$

Supposing that $\xi_0 \neq 0$ and using the mapping (2.7), we obtain the affine algebraic variety

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[3] However, a curve C in $\mathbf{P}^3$ may always be projected in $\mathbf{P}^2$ in such a way that the resulting plane algebraic curve has only ordinary double points.

in $\mathbf{C}^3$ associated to the polynomial equations

$$f(x_1, x_2, x_3) = g(x_1, x_2, x_3) = 0. \tag{2.25}$$

Apart from the exceptional cases described in [4], the above relations describe the complete intersections of two hypersurfaces in $\mathbf{C}^3$. One may also view $f(x)$ and $g(x)$ as polynomials of degrees $m \leq d_F$ and $n \leq d_G$, respectively in the variable x_3. Eliminating the latter, the resultant $R(x_1, x_2)$ is a polynomial in x_1 and x_2 of degree mn and the equation

$$R(x_1, x_2) = 0 \tag{2.26}$$

represents the projection of the curve (2.25) on $\mathbf{C}^2$. Let us notice that with respect to the case of plane curve, this projection is not unique. For instance, it is possible to eliminate from (2.25) the variable x_2 instead of x_3. In the latter case, one obtains a different resultant $R'(x_1, x_3)$ and a different projection onto $\mathbf{C}^2$:

$$R'(x_1, x_3) = 0. \tag{2.27}$$

To (2.25) one can also associate a compact Riemann surface Σ_g of genus g constructed as a ramified covering of $\mathbf{P}^1$. For example, in the neighborhood of a point where the Jacobian

$$J^1(x) = \frac{\partial f(x)}{\partial x_2} \frac{\partial g(x)}{\partial x_3} - \frac{\partial g(x)}{\partial x_2} \frac{\partial f(x)}{\partial x_3} \tag{2.28}$$

is different from zero, $x_1 \in \mathbf{P}^1$ becomes a good local coordinate and it is possible to solve the system of algebraic equations (2.25) with respect to x_2 and x_3. In this way, one obtains two multivalued functions $x_2(x_1)$ and $x_3(x_1)$, whose analytic continuation on the complex line $\mathbf{P}^1$ defines a Riemann surface. The ramification points are those in which the condition $J^1(x) = 0$ is satisfied. We stress again the difference with respect to plane curves, because now there are two possible representations of Σ_g in terms of branched covers of $\mathbf{P}^1$, corresponding to Eqs. (2.26) and (2.27).

Finally, we show that Eq. (2.25) includes the branched covers of Riemann surface as a particular sub-case. Indeed, let us consider a compact Riemann surface represented as a branched cover of $\mathbf{P}^1$ associated to the vanishing of a polynomial

$$f(x_1, x_2) = 0. \tag{2.29}$$

Any Riemann surface of genus g can be mapped into a branched cover of this kind in a limited region of its moduli space. Solving Eq. (2.29) with respect to x_2, the Riemann surface Σ_g is parameterized as a curve $\mathbf{C}^2$ with coordinates $(x_1, x_2(x_1))$. Starting from such coordinates it is still possible to realize a branched cover $\tilde{\Sigma}$ of Σ_g by requiring that

$$g(x_2, x_3) = 0. \tag{2.30}$$

3. The Weierstrass kernel

The Cauchy kernel

$$K_{\mathrm{C}}(x_1, x_2) = \frac{\mathrm{d}x_1}{x_1 - x_2} \tag{3.1}$$

plays a fundamental role in the construction of the amplitudes of conformal field theories on the complex plane $\mathbf{C}$. This kernel has the following two properties [3]:

1. As a function of x_1, the kernel is a meromorphic differential with two simple poles in $x_1 = x_2$ and in $x_1 = \infty$. The residues are $+1$ and -1, respectively.
2. As a function of x_2, the kernel is a meromorphic function with a simple pole in $x_2 = x_1$ and a single zero in $x_2 = \infty$.

A meromorphic function with a single pole cannot exist on a compact two-dimensional manifold. For this reason, usually it is only required that an analog $K(p, q)\,\mathrm{d}p$ of the Cauchy kernel on a Riemann surface Σ_g should have the following asymptotic behavior in a neighborhood of the point q [3]:

$$K(p, q) \sim \frac{\mathrm{d}p}{p - q} + A(p, q), \tag{3.2}$$

where $A(p, q)$ is finite at $p = q$.

A kernel with the above property can be constructed on n-sheeted coverings of $\mathbf{P}^1$ using a well-known algorithm of Weierstrass [53]. To this purpose, let us consider a Riemann surface Σ_g represented as the locus of points defined by Eq. (2.17). Solving this equation for x_2, one obtains a multivalued function $x_2(x_1)$, while x_1 is a d-degree mapping $x_1 : \Sigma_g \to \mathbf{P}_1$ from the Riemann surface to the projective sphere $\mathbf{P}^1$. Any point $p \in \Sigma_g$ is in a 1–1 correspondence with a point on the branched cover $x(p) \equiv x_1(p), x_2(x_1(p))$.

Let us put now $y(q) \equiv x_1(q), x_2(x_1(q))$, where $q \neq p$. Then an analog of the Cauchy kernel on the branched cover is the following Weierstrass kernel:

$$K_{\mathrm{W}}(x, y) = \frac{f(y_1, x_2)}{(x_2 - y_2)f^{(0,1)}(x_1, x_2)} \frac{\mathrm{d}x_1}{x_1 - y_1}. \tag{3.3}$$

To study its behavior in the limit $x_1 \to y_1$, one can expand $f(y_1, x_2)$ in series of Taylor near the point $x_2 = y_2$:

$$f(y_1, x_2) \sim f(y_1, y_2) + f^{(0,1)}(y_1, y_2)(x_2 - y_2) + \cdots. \tag{3.4}$$

Since $f(y_1, y_2) = 0$, it is clear that the Weierstrass kernel satisfies requirement (3.2)

$$K_{\mathrm{W}}(x, y) \sim \frac{\mathrm{d}x_1}{x_1 - y_1}. \tag{3.5}$$

The next term in the Taylor expansion gives in fact a contribution proportional to

$$A(x_1, y_1) = \frac{f^{(0,2)}(y_1, y_2)f^{(1,0)}(x_1, x_2)}{f^{(0,1)}(x_1, x_2)}\,\mathrm{d}x_1, \tag{3.6}$$

which is finite when $x_1 = y_1$.

One should also consider those points $p \neq q \in \Sigma_g$ which, on the branched cover, corre-spond to coordinates $x(p)$ and $y(q)$ such that $x_1(p) = x_1(q)$ and $x_2(x_1(p)) \neq x_2(x_1(q))$. In this case, there is no spurious pole in the Weierstrass kernel despite the presence of the

factor $(x_1 - y_1)^{-1}$. As a matter of fact, expanding the function $f(y_1, x_2)$ in series of Taylor in y_1:

$$f(y_1, x_2) = f(x_1, x_2) + f^{(1,0)}(x_1, x_2)(x_1 - y_1) + \cdots , \tag{3.7}$$

and substituting in (3.3), one obtains

$$K_W(x, y) \sim \frac{f^{(1,0)}(x_1, x_2)}{(x_2 - y_2) f^{(0,1)}(x_1, x_2)} \, dx_1, \tag{3.8}$$

which is finite because $x_2 - y_2 \neq 0$ by hypothesis.

One may also check that the Weierstrass kernel has no spurious poles at the branch points if the curve is regular. To this purpose it is possible to exploit the relation

$$\frac{dx_1}{f^{(0,1)}(x_1, x_2)} = \frac{dx_2}{f^{(1,0)}(x_1, x_2)}, \tag{3.9}$$

which is a consequence of the implicit function theorem [85]. Let us now consider a branch point $x_1 = x_1(a)$ of *multiplicity* v, where v is an integer and $a \in \Sigma_g$. We suppose for the moment that $x_2(x_1(a))$ is finite. Near the branch point we choose a good local coordinate t such that

$$x_1 - x_1(a) = t^v. \tag{3.10}$$

Since $x_2(x_1(a))$ is not divergent, its approximate expansion in powers of t will look as follows:

$$x_2 \sim \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots \tag{3.11}$$

with $\alpha_{0,1,2}$ being constants. Thus $dx_1 \sim v t^{v-1} \, dt$ and $dx_2 \sim dt$. Remembering that the function $f^{(1,0)}(x_1, x_2)$ does not vanish at a branch point due to the regularity hypothesis (2.4), we find that near $x_1(a)$, Eq. (3.9) is approximated by

$$\frac{dx_1}{f^{(0,1)}(x_1, x_2)} \sim dt. \tag{3.12}$$

This shows that the zeros of $f^{(0,1)}(x_1, x_2)$ are absorbed by the corresponding zeros of the differential dx_1, so that the Weierstrass kernel cannot be singular at the finite branch points. If x_2 has a pole of order k near a branch point, instead, it is always possible to perform the change of variables

$$x_2' = (x_1 - x_1(a))^k x_2. \tag{3.13}$$

In the new coordinates, x_2' remains finite at the branch point $x_1(a)$ and the above demonstration applies again.

Of course, on a Riemann surface a single pole is not allowed, so that the Weierstrass kernel (3.3) must contain also other spurious poles both as a function of x_1 and y_1. Typically, they appear whenever the variables x_1, x_2 and y_1, y_2 become infinitely large. A general procedure to subtract these spurious poles while keeping the property (3.2) has been already discussed

in Refs. [65,66]. The amplitudes of some free conformal field theories on branched covers of $\mathbf{P}^1$ have been constructed in.

To conclude this section, we present a beautiful formula of Weierstrass [53] to construct a second kind differential on an hyperelliptic curve. Thus, we consider curves of the kind (2.21), where

$$P_{2g+2}(x_1) = A_0 + A_1 x_1 + \cdots + A_{2g+2} x^{2g+2}. \tag{3.14}$$

Let us define the function:

$$\begin{aligned} R(x_1, x_1') &= A_0 + \tfrac{1}{2} A_1 (x_1 + x_1') + A_2 x_1 x_1' + \tfrac{1}{2} A_3 (x_1 + x_1') x_1 x_1' + A_4 (x_1 x_1')^2 \\ &\quad + \tfrac{1}{2} A_5 (x_1 + x_1')(x_1 x_1')^2 + \cdots + \tfrac{1}{2} A_{2g+1}(x_1 + x_1')(x_1 x_1')^g \\ &\quad + A_{2g+2}(x_1 x_1')^{g+1}. \end{aligned} \tag{3.15}$$

Clearly, $R(x_1, x_1')$ satisfies the property

$$R(x_1, x_1') = R(x_1', x_1). \tag{3.16}$$

Moreover, if $x_1 = x_1'$, one has that

$$R(x_1, x_1') = P_{2g+2}(x_1), \qquad \left. \frac{\partial R(x_1, x_1')}{\partial x_1} \right|_{x_1 = x_1'} = \frac{1}{2} \frac{\partial P_{2g+2}(x_1)}{\partial x_1}. \tag{3.17}$$

The differential of the second kind is given by

$$\tau_{x_1'}(x_1) = -\frac{y(x_1) y'(x_1) + R(x_1, x_1')}{2(x_1 - x_1')^2 y(x_1) y'(x_1)} \, dx_1, \tag{3.18}$$

where $y'(x_1) = \partial y(x_1)/\partial x_1$. It is easy to show using the properties (3.16) and (3.17) that $\tau_{x_1'}(x_1)$ has only a pole of the second order in $x_1 = x_1'$ as it should be. Unfortunately, it is not simple to extend the elegant formula (3.18) to the Z_n symmetric curves, not to mention the general plane algebraic curves of Eq. (2.17) or the even more complicated curves (2.25).

4. Generalized Weierstrass kernels

In this section, we construct analogs of the Weierstrass kernel on affine algebraic curves defined by the system of equations (2.25). Even if it will not be strictly necessary, we suppose to fix the ideas that the intersection of the two hypersurfaces in (2.25) is complete and gives as a result an algebraic curve C which coincides, modulo conformal transformations, with a Riemann surface Σ.

The following two different cases can formally be treated in the same way. On one side, x_1, x_2, x_3 may be interpreted as coordinates in $\mathbf{C}^3$, so that Σ is not compact due to the absence of the points at infinity. Alternatively, the vanishing of the polynomials (2.25) can be associated to a ramified covering of $\mathbf{P}^1$ as we have seen in Section 2 and Σ is a compact Riemann surface of genus g.

Let us consider as in the previous section two different points $p, q \in \Sigma$. On the algebraic curve C they correspond to coordinates $x(p) = x_1(p), x_2(p), x_3(p)$ and $y(q) = y_1(q), y_2(q), y_3(q)$. A possible analog of the Weierstrass kernel on C is given by

$$K_{\mathrm{sym}}(x, y) = \frac{1}{3} \sum_{i=1}^{3} \frac{N^i(x, y)}{J^i(x)} \frac{\mathrm{d}x_i}{(x_1 - y_1)(x_2 - y_2)(x_3 - y_3)}, \qquad (4.1)$$

where

$$J^i(x) = \epsilon^{ikl} \frac{\partial f(x)}{\partial x_k} \frac{\partial g(x)}{\partial x_l}, \qquad (4.2)$$

$$N^1(x, y) = f(y_1, y_2, x_3)g(x_1, y_2, x_3) - f(x_1, y_2, x_3)g(y_1, y_2, x_3), \qquad (4.3)$$

$$N^2(x, y) = f(x_1, y_2, y_3)g(x_1, x_2, y_3) - f(x_1, x_2, y_3)g(x_1, y_2, y_3), \qquad (4.4)$$

$$N^3(x, y) = f(y_1, x_2, y_3)g(y_1, x_2, x_3) - f(y_1, x_2, x_3)g(y_1, x_2, y_3). \qquad (4.5)$$

Here ϵ^{ikl} denotes the completely antisymmetric tensor in three dimensions with the convention $\epsilon^{123} = 1$. We note that the variables x, y and the functions f and g enter symmetrically in the expression of the kernel (4.1), as it should be since none of them plays a privileged role in the definition of the algebraic curve. The symmetry under the exchange of f and g is also related to the freedom of projecting the curve in the two possible ways shown by Eqs. (2.26) and (2.27).

Equivalent kernels can be obtained starting from $K_{\mathrm{sym}}(x, y)$ and adding differentials in such a way that the behavior near the singularity in $x = y$ remains unchanged. For instance, exploiting the identities:

$$\frac{\mathrm{d}x_1}{J^1} = \frac{\mathrm{d}x_2}{J^2} = \frac{\mathrm{d}x_3}{J^3}, \qquad (4.6)$$

which are a consequence of the implicit function theorem and using the fact that the numerators N^1, N^2, and N^3 differ each other by functions that vanish in $x = y$, one may derive the following kernel:

$$K(x, y) = \frac{N^1(x, y)}{J^1(x)} \frac{\mathrm{d}x_1}{(x_1 - y_1)(x_2 - y_2)(x_3 - y_3)}. \qquad (4.7)$$

The above kernel is less symmetric than $K_{\mathrm{sym}}(x, y)$, but has a more compact expression. In the particular case of a branched cover of a plane curve, in which the algebraic equations (2.25) assume the form (2.29) and (2.30), $K(x, y)$ is simply given by

$$K(x, y) = -\frac{f(x_1, y_2)g(y_2, x_3)}{[\partial g(x)/\partial x_3][\partial f(x)/\partial x_2]} \frac{1}{(x_1 - y_1)(x_2 - y_2)(x_3 - y_3)}. \qquad (4.8)$$

Let us investigate the behavior of $K(x, y)$ near $x = y$. To this purpose, it is sufficient to expand $N^1(x, y)$ with respect to y at the point x. At the leading order

$$N^1(x, y) \sim J^1(x)\Delta x_2 \Delta x_3, \qquad (4.9)$$

where $\Delta x_i = y_i - x_i$ is very small by hypothesis. We note that, in principle, there are also contributions proportional to Δx_1 in the expansion of $N^1(x, y)$. In order to obtain Eq. (4.9),

Δx_1 has been expressed in terms of Δx_2 and Δx_2 with the help of (4.6). Substituting (4.9) in (4.7), we find that $K(x, y)$ satisfies the property (3.2) as desired

$$K(x, y) \sim \frac{\mathrm{d}x_1}{x_1 - y_1}. \tag{4.10}$$

A similar calculation for $K_{\mathrm{sym}}(x, y)$ gives the result

$$K_{\mathrm{sym}}(x, y) \sim \frac{1}{3} \sum_{i=1}^{3} \frac{\mathrm{d}x_i}{x_i - y_i}. \tag{4.11}$$

In this case one should remember that not all variables x_1, x_2, x_3 are independent due to the relations (2.25), so that Eq. (4.11) must be worked out further. Assuming for instance that x_1 is a good local coordinate in a neighborhood of the point p, one can solve the system of algebraic equations (2.25) with respect to the remaining variables, so that $x_j = x_j(x_1)$ for $j = 2, 3$. In the same way $y_j = x_j(y_1)$ for $j = 2, 3$ in a neighborhood of q, with $y_1 = x_1 + \Delta x_1$. As a consequence

$$\mathrm{d}x_j(x_1) = \frac{\mathrm{d}x_j}{\mathrm{d}x_1} \mathrm{d}x_1, \quad x_j(y_1) - x_j(x_1) \sim -\frac{\mathrm{d}x_j}{\mathrm{d}x_1} \Delta x_1, \quad j = 2, 3. \tag{4.12}$$

Using the above relations in (4.11), one obtains

$$K_{\mathrm{sym}}(x, y) \sim \frac{\mathrm{d}x_1}{x_1 - y_1}. \tag{4.13}$$

Thus, also the kernel (4.1) has the requested behavior near the pole in $x = y$.

Besides the required simple pole in $x = y$, the kernels (4.1) and (4.7) have also spurious poles, which have to be controlled and suitably subtracted in order to construct physical correlation functions with desired singularities. The study of these spurious poles will be the subject of the rest of this section.

Since the structure of the kernels (4.1) and (4.7) consists in ratios of polynomials of x and y, their possible divergences may only occur at the zeros of the denominators $J^i(x)(x_1 - y_1)(x_2 - y_2)(x_3 - y_3)$ or at the infinities of the numerators $N^i(x, y)$, $i = 1, 2, 3$.

First of all, we consider the zeros of $J^1(x)$. The cases in which $i = 2, 3$ can be treated in an analogous way. Let $x(a) = (x_1(a), x_2(a), x_3(a))$, $a \in \Sigma$, be a point on C for which $J^1(x) = 0$. In $x(a)$ the system of algebraic equations (2.25) becomes no longer invertible with respect to x_2 and x_3. Given a good local coordinate t in a neighborhood of $x(a)$, this implies that

$$x_1 - x_1(a) = t^{\lambda}, \tag{4.14}$$

$$x_2 = \alpha_0 + \alpha_1 t^{\mu} + \cdots, \qquad x_3 = \beta_0 + \beta_1 t^{\nu} + \cdots, \tag{4.15}$$

where λ is an integer containing the integers μ and ν as sub-factors. In Eq. (4.15), we suppose that x_2 and x_3 do not diverge in a, so that λ, μ, ν are all positive. If not, it is always possible to perform in Eq. (2.25) a change of variables $x_2, x_3 \to x_2', x_3'$ similar to that of Eq. (3.13) to make the new variables x_2', x_3' finite in a. We note that the monodromy

properties of x_2 and x_3 may be in general different, so that μ and ν need not to be equal. Near $x(a)$ the relations (4.6) become

$$\frac{\lambda t^{\lambda-1}\,\mathrm{d}t}{J^1(x)} = \frac{\mu t^{\mu-1}\,\mathrm{d}t}{J^2(x)} = \frac{\nu t^{\nu-1}\,\mathrm{d}t}{J^3(x)}, \tag{4.16}$$

where x is a function of t given by Eqs. (4.14) and (4.15). At this point it is possible to invoke the regularity condition (2.4), which assures that $J^2(x)$, $J^3(x) \neq 0$ in $x = x(a)$. Since $\mu, \nu \leq \lambda$ it is easy to see from (4.16) that the zeros of $J^1(x)$ are absorbed by the corresponding zeros of the differential $\mathrm{d}x_1$. As a consequence, there are no spurious divergences at these points.

One can also verify that there are no poles when $(x_1 - y_1)(x_2 - y_2)(x_3 - y_3) = 0$ apart from the one in $x(p) = y(q)$, which is related to the required singularity in $p = q$. Spurious poles of this kind may in principle occur if two different points $p, q \in \Sigma$ correspond on the algebraic curve to coordinates $x(p)$ and $x(q)$ characterized by the fact that some of their components, but not all, coincide (for instance $x_1 = y_1$, $x_2 = y_2$ and $x_3 \neq y_3$). The proof that no spurious divergence arises in this case is straightforward and will not be reported here.

In conclusion, the kernels (4.1) and (4.7) diverge only at the poles of the numerators $N^i(x, y)$. In general, the latter are located at the points in which the variables x and y become very large.

5. The case of general non-hyperelliptic curves of genus four

A general non-hyperelliptic algebraic curve of genus four is given by the complete intersection of a quadric with a cubic as mentioned in Section 2. In fact, putting $d_F = 3$ and $d_G = 2$ in (2.24), one obtains exactly $g = 4$. For instance, one can choose in Eq. (2.23) $G(\xi)$ as follows:

$$G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_0\xi_1 - \xi_2\xi_3, \tag{5.1}$$

while $F(\xi)$ is a homogeneous polynomial of degree 3. In affine coordinates $\xi_i = \xi_i/\xi_0$, $i = 1, 2, 3$, $F(\xi)$ and $G(\xi)$ are replaced by

$$G(1, x_1, x_2, x_3) = g(x_1, x_2, x_3) = x_2x_3 - x_1, \tag{5.2}$$

$$F(1, x_1, x_2, x_3) = f(x_1, x_2, x_3) = x_3^3 + h_1(x_1, x_2)x_3^2 + h_2(x_1, x_2)x_3 + h_3(x_1, x_2), \tag{5.3}$$

where

$$h_i(x_1, x_2) = \sum_{\substack{k,l=0 \\ k+l\leq 3}}^{i} a_{kl}^{(i)} x_1^k x_2^l, \quad i = 1, 2, 3, \tag{5.4}$$

and the $a_{kl}^{(i)}$ are complex coefficients. We note that the polynomial $f(\xi_1, \xi_2, \xi_3)$ has been ordered according to the different powers of x_3. This is just a convention which does not

reflect any special role of x_3. In the same way one could order $f(\xi_1, \xi_2, \xi_3)$ with respect to the powers of x_1 or x_2. All necessary ingredients to construct the Weierstrass kernels (4.1) and (4.7) are derived in a straightforward way substituting Eqs. (5.2) and (5.3) in (4.2)–(4.5).

In the following, we will assume that x_1 is a good local coordinate of the curve. The cases in which $x_1 \in \mathbf{C}$ or $x_1 \in \mathbf{P}_1$ can be formally treated in the same way. Accordingly, we solve the system of algebraic equations

$$f(x_1, x_2, x_3) = g(x_1, x_2, x_3) = 0 \tag{5.5}$$

with respect to x_1.[4] As a result, one obtains two multivalued functions $x_2(x_1)$ and $x_3(x_1)$. Due to Eq. (5.2), both $x_2(x_1)$ and $x_3(x_1)$ share the same monodromy properties. They define a Riemann surface Σ_4 constructed in terms of sheets. It is easy to check that $x_2(x_1)$ and $x_3(x_1)$ have six branches, so that Σ_4 consists of six sheets glued together at the branch lines. The computation of the resultant $R(x_1, x_2)$ of the two algebraic equations (5.5) is straightforward and gives the following plane curve equation associated to Σ_4:

$$x_1^3 + h_1(x_1, x_2)x_1^2 x_2 + h_2(x_1, x_2)x_1 x_2^2 + h_3(x_1, x_2)x_2^3 = 0, \tag{5.6}$$

which indeed describes a curve of genus four.

Due to the peculiar role played by the variable x_1, it is natural to consider the kernel $K(x, y)$ instead of the more symmetric one of Eq. (4.1). Inserting Eqs. (5.2) and (5.3) in (4.7), one obtains

$$K(x, y) = -\frac{dx_1}{(x_1 - y_1)J^1(x)}\left[\frac{x_3 f(y_1, y_2, y_3)}{x_3 - y_3} + \frac{y_2 f(x_1, y_2, y_3)}{x_2 - y_2}\right], \tag{5.7}$$

where we should remember that we are dealing with multivalued functions $x_j = x_j(x_1)$ and $y_j = y_j(y_1)$ for $j = 2, 3$. With respect to the general formula (4.7), in this case some simplifications have been possible because of the particular form of $g(x_1, x_2, x_3)$. One can check that the above kernel has the desired pole when $x_i = y_i$ for $i = 1, 2, 3$. If one wishes to study the kernel (5.7) in the neighborhood of a branch point where $J^1(x) = 0$, it is possible to perform the conformal transformation $x_1 = x_1(x_2)$.

Let us now concentrate on the spurious divergences of $K(x, y)$. From the previous section, we know that they may only occur at the infinities of the variables $x_i, y_i, i = 1, 2, 3$. In this case, the situation is made simpler by the fact that $x_2(x_1)$ and $x_3(x_1)$ have no poles for finite values of x_1. To show that, let us imagine that a point $a \in \Sigma_4$ corresponds on the algebraic curve to a point $x_1(a)$ where x_3 has a pole of order s:

$$x_3(x_1) \sim (x_1 - x_1(a))^{-s}. \tag{5.8}$$

Due to Eq. (5.2), the function $x_2(x_1)$ has a zero of the same order at the same point

$$x_2(x_1) \sim (x_1 - x_1(a))^{s}. \tag{5.9}$$

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[4] There is no loss of generality in doing that. If one wishes to study the algebraic curve in the neighborhood of a branch point, where x_1 is no longer a good coordinate, it is always possible to perform a conformal transformation and to consider x_1 as a function of x_2 or x_3.

Thus, in the limit $x_1 = x_1(a)$ the polynomials $h_i(x_1, x_2)$, $i = 1, 2, 3$, may be replaced by suitable constants A_1, A_2, A_3 neglecting higher order terms in $x_1 - x_1(a)$. Hence, Eq. (5.3) is approximated by

$$x_3^3 + A_1 x_3^2 + A_2 x_3 + A_3 = 0. \tag{5.10}$$

Clearly, the above equation has no solutions if $x_3 = \infty$. Analogously, since x_3 and x_2 enter in Eq. (5.5) symmetrically, it is possible to verify that $x_2(x_1)$ has no divergences for finite values of x_1.

To study the singularities of $x_2(x_1)$ and $x_3(x_1)$ at infinity, it is convenient to introduce the new variable $x_1' = x_1^{-1}$. Let us now suppose that x_2 and x_3 have the following behavior near $x_1' = 0$:

$$x_2 = \alpha (x_1')^s + \cdots, \qquad x_3 = \beta (x_1')^{-1-s} + \cdots. \tag{5.11}$$

The second relation (5.11) is again a consequence of (5.2). Substituting the ansatz (5.11) in (5.3), it is easy to verify that the latter equation is satisfied only if $s = -1$ or 0. In the first case, there are three branches of x_2 and x_3 such that x_2 diverges

$$x_2 \sim \frac{1}{x_1'}, \qquad x_3 \sim \text{const.} \tag{5.12}$$

If $s = 0$, instead, there are other three branches in which x_3 becomes singular

$$x_2 \sim \text{const.}, \qquad x_3 \sim \frac{1}{x_1'}. \tag{5.13}$$

At this point, we are ready to discuss the spurious poles of the kernel (5.7). As a meromorphic differential in x_1, $K(x, y)$ has three simple poles in $x_1 = 0$. The latter occur in the three branches of x_2 and x_3 where Eq. (5.13) is satisfied. In each of these branches, $K(x, y)$ has residue $-\frac{1}{3}$:

$$K(x, y) = -\frac{1}{3} \frac{\mathrm{d}x_1'}{x_1'} + \cdots. \tag{5.14}$$

It is easy to check that there are no other spurious singularities in x.[5] Thus $K(x, y)$ is a differential of the third kind on Σ_4. Taking into account also the simple pole in $x = y$, the sum of all its residues vanishes as it should be on a compact surface.

To study the singularities with respect to the variables y, it is convenient to rewrite $K(x, y)$ in a slightly different form, obtained by expanding in (5.7) $f(y_1, y_2, x_3)$ and $f(x_1, y_2, x_3)$ in powers of x_3 and y_2, respectively. Since $f(x)$ is a polynomial in its arguments of degree 3, the expansions below

$$f(y_1, y_2, x_3) = \sum_{n=1}^{3} \frac{\partial^n f(y)}{\partial y_3^n} \frac{(x_3 - y_3)^n}{n!}, \tag{5.15}$$

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[5] In principle, one would expect the appearance of singularities also in the branches in which Eq. (5.12) is satisfied due to the symmetry between the variables x_2 and x_3 in (5.5). However, we remember that this symmetry has been explicitly broken by the way in which the kernel $k(x, y)$ has been constructed.

$$f(x_1, y_2, x_3) = \sum_{n=1}^{3} \frac{\partial^n f(x)}{\partial x_2^n} \frac{(x_2 - y_2)^n}{n!} \tag{5.16}$$

are exact. As a consequence, inserting (5.15) and (5.16) in (5.7), we have that

$$K(x, y) = \frac{P(x, y)}{J^1(x)(x_1 - y_1)} \, dx_1, \tag{5.17}$$

where

$$P(x, y) = \sum_{n=1}^{3} \left[y_2 \frac{\partial^n f(x)}{\partial x_2^n} \frac{(x_2 - y_2)^{n-1}}{n!} - x_3 \frac{\partial^n f(y)}{\partial y_3^n} \frac{(x_3 - y_3)^{n-1}}{n!} \right]. \tag{5.18}$$

Now, we exploit the fact that the spurious divergences of $K(x, y)$ are located at the points in which $y_1 = \infty$ as the previous analysis has shown. Therefore, it is convenient to keep in the kernel only the contributions which diverge in $y_1 = \infty$. Using the formula $1/(x_1 - y_1) = -(1/y_1)\sum_{n=0}^{\infty}(x_1/y_1)^n$, we find

$$K(x, y) \sim \frac{dx_1}{J^1(x)}[\mathcal{A}_1(y) + x_1\mathcal{A}_2(y) + x_2\mathcal{A}_3(y) + x_3\mathcal{A}_4(y)], \tag{5.19}$$

where

$$\mathcal{A}_1(y) = -y_1^{-1}(a_{03}^{(3)} y_2^3 + a_{02}^{(3)} y_2^2), \tag{5.20}$$

$$\mathcal{A}_2(y) = -a_{03}^{(3)} \frac{y_2^3}{y_1^2} - a_{12}^{(3)} \frac{y_2^2}{y_1}, \tag{5.21}$$

$$\mathcal{A}_3(y) = -a_{03}^{(3)} \frac{y_2^2}{y_1}, \tag{5.22}$$

$$\mathcal{A}_4(y) = y_1^{-1}[-a_{02}^{(2)} y_2^2 + 7y_3^2 + 3y_3 h_1(y_1, y_2) + h_2(y_1, y_2)]. \tag{5.23}$$

We notice that $K(x, y)$ has the following behavior near the spurious poles in $y_1 = \infty$:

$$K(x, y) \sim \sum_{i=1}^{4} \omega_i(x) \mathcal{A}_i(y), \tag{5.24}$$

where the $\omega_i(x)$ are holomorphic differentials. As a matter of fact, using Eqs. (5.12) and (5.13) it is easy to check that a basis of holomorphic differentials on Σ_4 is

$$\omega_1(x) = \frac{dx_1}{J^1(x)}, \qquad \omega_2(x) = \frac{x_1 \, dx_1}{J^1(x)}, \tag{5.25}$$

$$\omega_3(x) = \frac{x_2 \, dx_1}{J^1(x)}, \qquad \omega_4(x) = \frac{x_3 \, dx_1}{J^1(x)}. \tag{5.26}$$

As a result, it is possible to conclude that the divergent part of $K(x, y)$ given in Eq. (5.19) is proportional to holomorphic differentials in x. This property of the kernel (5.7) will be crucial in subtracting the spurious divergences from the amplitudes of the conformal field theories which will be the subject of the next section.

6. Free conformal field theories on a general non-hyperelliptic curve of genus four

In this section, we apply the previous results to the computation of the amplitudes of the free conformal field theories appearing in the action of bosonic strings on a general non-hyperelliptic surface of genus four Σ_4. These systems are well known and represent a good way to test the generalized Weierstrass differential constructed in Section 4. Apart from the above mentioned works in the case of hyperelliptic and non-hyperelliptic curves, they have been studied by various authors and different methods on Riemann surfaces (see, for instance [86–115]).

First of all, we discuss the case of fermionic b–c systems with integer spin $\lambda = 1, 2$. In isothermal coordinates p, $\bar{p}$ where the metric becomes conformally flat, their action is given by

$$S_{bc} = \int_{\Sigma_4} \mathrm{d}^2 p \, b \bar{\partial} c, \tag{6.1}$$

where $\mathrm{d}^2 p = i \, \mathrm{d}p \wedge \mathrm{d}\bar{p}$ and $\bar{\partial} = \partial/\partial \bar{p}$. The fields b are meromorphic tensors on Σ_4 with λ lower indices, while the fields c are characterized by $\lambda - 1$ upper indices. From Eq. (6.1), one obtains the following equations of motion:

$$\bar{\partial} b = \bar{\partial} c = 0. \tag{6.2}$$

We start with the case $\lambda = 2$. We denote with ϕ_μ, $\mu = 1, \ldots, 9$, the holomorphic quadratic differentials which represent a basis of non-trivial solutions of (6.2). If $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ are points in Σ_4, the non-vanishing correlation functions of the b–c systems

$$\mathcal{G}_2(p; q) = \left\langle \prod_{\alpha=1}^{m} b(p_\alpha) \prod_{\beta=1}^{n} c(q_\alpha) \right\rangle \tag{6.3}$$

should satisfy the relation $m - n = 3g - 3 = 9$.

The zeros and poles of $\mathcal{G}_2(p; q)$ are determined by the physical properties of the fields. To specify their locations and orders it is convenient to introduce the concept of *divisor*. Let $\Delta[T]$ denote the divisor of a given meromorphic tensor $T(p)$ on the Riemann surface. If T has zeros at $p = p_i$ of order μ_i, $i = 1, \ldots, m_{\text{zeros}}$ and poles of order ν_j at $p = q_j$, $j = 1, \ldots, n_{\text{poles}}$, then $\Delta[T]$ can be written as follows:

$$\Delta[T] = \sum_{i=1}^{m_{\text{zeros}}} \mu_i p_i - \sum_{j=1}^{n_{\text{poles}}} \nu_j q_j. \tag{6.4}$$

The correlation function (6.3) is a tensor with two lower indices in each variable p_α, $\alpha = 1, \ldots, m$ and the following divisor:

$$\Delta_{p_\alpha}[\mathcal{G}_2] = \sum_{\substack{\alpha'=1 \\ \alpha' \neq \alpha}}^{m} p_{\alpha'} - \sum_{\substack{\beta'=1 \\ \beta' \neq \beta}}^{n} q_{\beta'}. \tag{6.5}$$

With respect to q_β, instead, $\mathcal{G}_2(p; q)$ is a vector with an upper index and divisor

$$\Delta_{q_\beta}[\mathcal{G}_2] = \sum_{\substack{\beta'=1 \\ \beta' \neq \beta}}^{n} q_{\beta'} - \sum_{\substack{\alpha'=1 \\ \alpha' \neq \alpha}}^{m} p_{\alpha'}. \tag{6.6}$$

To construct $\mathcal{G}_2(p; q)$ explicitly, we represent Σ_4 as a ramified covering C_4 associated to the system of algebraic equations (5.5) treated in the previous section. We begin by noting that a meromorphic tensor $T(p)$ with λ indices on the Riemann surface Σ_4 corresponds in C_4 to a tensor $T(x(p)) = T_{x_1,...,x_1}(x(p)) \, dx_1^\lambda$, where $x(p) = x_1(p), x_2(p), x_3(p)$ and $x_1(p) \in \mathbf{P}_1$. A pole or a zero of $T(p)$ in $p = q$ corresponds to a pole or a zero of $T(x(p))$ of the same order in $x(p) = y(q)$.

A basis of independent holomorphic quadratic differentials is given by

$$\phi_1(x) = \left(\frac{dx_1}{J^1(x)} \right)^2, \quad \phi_2(x) = x_1 \left(\frac{dx_1}{J^1(x)} \right)^2, \quad \phi_3(x) = x_2 \left(\frac{dx_1}{J^1(x)} \right)^2, \tag{6.7}$$

$$\phi_4(x) = x_3 \left(\frac{dx_1}{J^1(x)} \right)^2, \quad \phi_5(x) = \left(\frac{x_1 \, dx_1}{J^1(x)} \right)^2, \quad \phi_6(x) = \left(\frac{x_2 \, dx_1}{J^1(x)} \right)^2, \tag{6.8}$$

$$\phi_7(x) = \left(\frac{x_3 \, dx_1}{J^1(x)} \right)^2, \quad \phi_8(x) = x_1 x_2 \left(\frac{dx_1}{J^1(x)} \right)^2, \quad \phi_9(x) = x_1 x_3 \left(\frac{dx_1}{J^1(x)} \right)^2. \tag{6.9}$$

Clearly, a quadratic differential of the kind $\phi_{10}(x) = x_2 x_3 (dx_1/J^1(x))^2$ would not be independent due to Eq. (5.2).

It is also easy to check that the following quadratic differential:

$$K_2(x, y) = - \left(\frac{dx_1}{J^1(x)} \right)^2 \left[\frac{x_3 f(y_1, y_2, x_3)}{(x_1 - y_1)(x_3 - y_3)} + \frac{y_2 f(x_1, y_2, x_3)}{(x_1 - y_1)(x_2 - y_2)} \right] \tag{6.10}$$

has only a simple pole at the point $x_i = y_i$, $i = 1, 2, 3$. In fact, $K_2(x, y)$ is obtained multiplying together the Weierstrass kernel (5.7) and the holomorphic differential $\omega_1(x)$ of Eq. (5.26). The zeros of the latter cancel exactly the poles of $K(x, y)$. Indeed, since $J^1(x) = (\partial f/\partial x_2) - x_1(\partial f/\partial x_3)$ and f is a polynomial of degree 3 in x_3, we have that $J^1(x) \sim -(1/x'^3)$ at infinity in the three branches in which Eq. (5.13) is satisfied.

At this point we are ready to write the correlation functions of the b–c systems with $\lambda = 2$ on C_4:

$$\mathcal{G}_2(p; q)$$
$$= \det \begin{vmatrix} K_2(x(p_1), y(q_1)) & \cdots & K_2(x(p_1), y(q_1)) & \phi_1(x(p_1)) & \cdots & \phi_9(x(p_1)) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_2(x(p_m), y(q_1)) & \cdots & K_2(x(p_m), y(q_n)) & \phi_1(x(p_m)) & \cdots & \phi_9(x(p_m)) \end{vmatrix}. \tag{6.11}$$

Due to the properties of determinants, the right-hand side of the above equation has the desired simple zeros whenever $x_i(p_\alpha) = x_i(p_{\alpha'})$, $\alpha, \alpha' = 1, \ldots, m$ and $y_i(q_\beta) = y_i(q_{\beta'})$, $\beta, \beta' = 1, \ldots, n$, $i = 1, 2, 3$. Moreover, all the poles of $\mathcal{G}_2(p; q)$ are simple and occur at the points in which $x_i(p_\alpha) = y_i(q_\beta)$.

In principle, there could be also spurious poles due to the fact that $K_2(x, y)$ diverges when the variable y becomes very large. However, from what it has been discussed in the previous section, it is easy to realize that $K_2(x, y)$ has the following behavior in $y_1 = \infty$:

$$K_2(x, y) \sim \sum_{i=1}^{4} \omega_i(x) \mathcal{A}_i(y), \tag{6.12}$$

where the functions $\mathcal{A}_i$ have been defined in Eq. (5.24) and diverge when $y_1 \to \infty$. Terms of the form (6.12) consist in a linear combination of quadratic differentials, which does not contribute in the determinant of Eq. (6.11).

Let us now treat the case $\lambda = 1$. On the Riemann surface Σ_4 the non-vanishing correlations functions are given by

$$\mathcal{G}_1(p, q) = \left\langle \prod_{\alpha=1}^{m} b(p_\alpha) \prod_{\beta=1}^{n} c(q_\alpha) \right\rangle, \tag{6.13}$$

where $m - n = g - 1 = 3$. Again, the poles and zeros of the above correlator are determined by the physical properties of the b–c fields. The divisors of $\mathcal{G}_1(p, q)$ are similar to those of $\mathcal{G}_2(p, q)$ with simple zeros whenever $p_\alpha = p_{\alpha'}$ or $q_\beta = p_{\beta'}$ and simple poles when $p_\alpha = q_\beta$, with $\alpha, \alpha' = 1, \ldots, m$, $\beta, \beta' = 1, \ldots, n$.

The explicit construction of $\mathcal{G}_1(p; q)$ on the ramified covering C_4 goes as follows. First of all, we define the differential

$$v_{y(q)y(q')}(x(p)) = K(x(p), y(q)) - K(x(p), y(q')). \tag{6.14}$$

This is a differential of the third kind in $x(p)$, with two simple poles in $x(p) = y(q)$ and $x(p) = y(q')$ and residues $+1$ and -1, respectively. The spurious poles of $K(x(p), y(q))$ in $x_1 = \infty$ are canceled against the analogous poles of $K(x(p) - y(q'))$. At this point it is possible to write the expression of $\mathcal{G}_1(p; q)$:

$$\mathcal{G}_1(p; q)$$
$$= \det \begin{vmatrix} v_{y(q_1)y(q_n)}(x(p_1)) & \cdots & v_{y(q_{n-1})y(q_n)}(x(p_1)) & \omega_1(x(p_1)) & \cdots & \omega_4(x(p_1)) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{y(q_1)y(q_n)}(x(p_m)) & \cdots & v_{y(q_{n-1})y(q_n)}(x(p_m)) & \omega_1(x(p_m)) & \cdots & \omega_4(x(p_m)) \end{vmatrix}. \tag{6.15}$$

As in the case $\lambda = 2$, Eq. (5.24) implies the absence of spurious poles in the y variables in (6.15).

To conclude our list of conformal field theories which appear in bosonic string theory, we treat the scalar fields with action

$$S = \int_{\Sigma_4} \mathrm{d}^2 p \, \partial X \bar{\partial} X. \tag{6.16}$$

All the correlation functions of the scalar fields are obtained once the following correlator is known:

$$G(p; q, q') = \langle \partial_p X(p, \bar{p}) [X(q, \bar{q}) - X(q', \bar{q}')] \rangle \, \mathrm{d}p. \tag{6.17}$$

It turns out that $G(p; q, q')$ is a *canonical differential of the third kind* uniquely determined by the following properties:

1. $G(p; q, q')$ has only two simple poles in $p = q$ and $p = q'$ with residues $+1$ and -1, respectively.
2. The integral function $\int G(p; q, q')$ has purely imaginary periods when transported around the $2g = 8$ non-trivial homology cycles of Σ_4.

On the algebraic curve C_4 the Green function (6.17) can be written as a vector field $G(x(p); y(q), y(q'))$, where $x(p) = x_1(p), x_2(p), x_3(p)$, etc. $G(x(p); y(q), y(q'))$ coincides to the third kind differential $v_{y(q)y(q')}(x(p))$ defined in Eq. (6.14) up to a linear combination of holomorphic differentials, which is fixed by requirement (2). In practice, since it is hard to deal with integrals over homology cycles in the case of algebraic curves, it is convenient to formulate this requirement in terms of surface integrals. Indeed, it is possible to show that (2) is satisfied if and only if $G(x(p); y(q), y(q'))$ fulfills the following Riemann bilinear identities:

$$\int_{C_4} G(x(p); y(q), y(q')) \wedge \overline{\omega_i(x(p))} = 0, \quad i = 1, \ldots, 4. \tag{6.18}$$

The surface integrals over C_4 in (6.18) can be interpreted as integrals in a three-dimensional complex space concentrated in the solutions of Eq. (5.5) (see Appendix A). At this point we are able to write the Green function $G(x(p); y(q), y(q'))$ in terms of the third kind differential (6.14) and of the holomorphic differentials (5.25) and (5.26)

$$
\begin{aligned}
&G(x(p); y(q), y(q')) \\
&= \det \begin{vmatrix}
v_{y(q)y(q')}(x(p)) & \omega_1(x(p)) & \cdots & \omega_4(x(p)) \\
\int_{C_4} v_{y(q)y(q')} \wedge \overline{\omega_1} & \int_{C_4} \omega_1 \wedge \overline{\omega_1} & \cdots & \int_{C_4} \omega_1 \wedge \overline{\omega_4} \\
\vdots & \vdots & \ddots & \cdots \\
\int_{C_4} v_{y(q)y(q')} \wedge \overline{\omega_4} & \int_{C_4} \omega_4 \wedge \overline{\omega_1} & \cdots & \int_{C_4} \omega_4 \wedge \overline{\omega_4}
\end{vmatrix}.
\end{aligned} \tag{6.19}
$$

It is easy to see that the above differential of the third kind satisfies requirement (1) and the relations (6.18), which are equivalent to (2).

7. Conclusions

In Section 4, an analog of the Weierstrass kernel has been constructed on non-plane algebraic curves associated to the vanishing of two polynomials f and g. The freedom of adding linear combinations of differentials which do not change the behavior in $x = y$ has been exploited in order to get two different, but equivalent, versions of generalized

Weierstrass kernels. The first version $K_{\text{sym}}(x, y)$, given in (4.1), is symmetric with respect to the variables x, y and with respect to the exchange of f with g. This is in agreement with the fact that neither the coordinates nor the functions f and g play a special role in the equations which define the curve. The alternative kernel $K(x, y)$ of Eq. (4.7) has the advantage to have a more compact expression in comparison to $K_{\text{sym}}(x, y)$, but part of the symmetry under coordinate permutations is lost. If the branched cover of an algebraic curve is considered, the generalized Weierstrass kernel $K(x, y)$ has the simple form (4.8).

Furthermore, it has been verified that both kernels $K_{\text{sym}}(x, y)$ and $K(x, y)$ are third kind differentials with a simple pole in the desired point $x = y$ of the curve, in agreement with property (3.2), which characterizes the analogs of the Cauchy kernel on Riemann surfaces [3]. In Section 4, it has also been proved that spurious singularities may only occur at the points in which one or more of the components of the coordinates x or y approach infinity. In the absence of a general algorithm to treat these spurious singularities like that developed in the case of plane curves in [65,66], the terms to be subtracted in order to get the desired correlation functions should be derived separately for any given polynomials f and g. The example of a general non-hyperelliptic curve of genus four has been explicitly worked out and the amplitudes of bosonic string theory have been computed (see Eqs. (6.11), (6.15) and (6.19)).

Finally, nothing has been said about non-hyperelliptic Riemann surfaces obtained from the intersection of $n - 1$ hypersurfaces in $\mathbf{P}^n$ with $n > 3$. However, it is clear from Eqs. (4.1) and (4.7) what should be the structure of the generalized Weierstrass kernel on these curves. The Jacobians $J^i(x)$ should be replaced by analogous Jacobians containing derivatives of the $n - 1$ polynomials $f_1, \ldots, f_{n-1}$ with respect to any possible $(n - 1)$-dimensional subsets of the coordinates. The numerators $N^i(x, y)$ will contain a sum of products of polynomials $f_1, f_2, \ldots, f_{n-1}$, in which the dependence on the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is chosen in such a way that the spurious poles in the denominator given by the factor $\prod_{i=1}^{n}(x_i - y_i)$ are canceled, so that only the desired singularity in $x_i = y_i$, $i = 1, \ldots, n$ remains.

Acknowledgements

Appendix A. Surface integrals over C_4

In this section, we show that the surface integrals appearing in Eqs. (6.18) and (6.19) can be expressed as integrals in a three-dimensional complex space in the presence of Dirac δ-functions which impose the constraints (5.5). We suppose here that $x \in \mathbf{C}^3$, but the result is valid also for a compact curve, in which case $\mathbf{C}^3$ has to be replaced by $\mathbf{P}_1^3$ and the non-flat metric of $\mathbf{P}_1$ should be taken into account.

Let us consider a surface integral of the kind

$$I = \int_{C_4} \rho, \tag{A.1}$$

where $\rho(x(p))$ is a (1, 1)-form. In components $\rho(x(p)) = \rho_{x_1 \bar{x}_1}(x(p)) \, dx_1 \wedge d\bar{x}_1$ and

$$I = \int_{C_4} d^2 x_1(p) \rho_{x_1 \bar{x}_1}(x_1(p), x_2(p), x_3(p)). \tag{A.2}$$

Since C_4 is a curve associated to the system of algebraic equations (5.5), it is possible to rewrite the integral (A.1) as follows:

$$\int_{C_4} d^2 x_1(p) \rho_{x_1 \bar{x}_1}(x(p)) = \int_{\mathbf{C}^3} d^6 x \left| J \left(\begin{array}{cc} f & g \\ x_2 & x_3 \end{array} \right) \right|^2 \delta^{(4)}(f, g) \rho_{x_1 \bar{x}_1}(x(p)), \tag{A.3}$$

where $d^6 x$ is the volume element in $\mathbf{C}^3$,

$$J \left(\begin{array}{cc} f & g \\ x_2 & x_3 \end{array} \right) = \det \left| \begin{array}{cc} \dfrac{\partial f}{\partial x_2} & \dfrac{\partial g}{\partial x_2} \\[2mm] \dfrac{\partial f}{\partial x_3} & \dfrac{\partial g}{\partial x_3} \end{array} \right|, \tag{A.4}$$

and the Dirac δ-function $\delta^{(4)}(f, g)$ has been defined using the formulas of Gel'fand and Shilov [116]. After performing the change of variables $x_2, x_3 \rightarrow f, g$, this distribution becomes an usual four-dimensional δ-function

$$\delta^{(4)}(f, g) = \left(\frac{\partial_f \partial_{\bar{f}} + \partial_g \partial_{\bar{g}}}{2\pi^2} \right) \frac{1}{|f|^2 + |g|^2}. \tag{A.5}$$

Now let us apply to Eq. (A.3) the inverse transformation which brings back to the old coordinates. This can be done using the relations

$$\partial_f = \frac{1}{g_{x_2} f_{x_3} - g_{x_3} f_{x_2}} (g_{x_2} \partial_{x_3} - g_{x_3} \partial_{x_2}),$$

$$\partial_f = \frac{1}{g_{x_2} f_{x_3} - g_{x_3} f_{x_2}} (f_{x_3} \partial_{x_2} - f_{x_2} \partial_{x_3}). \tag{A.6}$$

Substituting the result in Eq. (A.3), we obtain an explicit expression of I in terms of three-dimensional complex integrals

$$\int_{C_4} d^2 x_1(p) \rho_{x_1 \bar{x}_1}(x(p))$$

$$= \frac{1}{2\pi^2} \int d^6 x \, \rho_{x_1 \bar{x}_1}(x(p)) [|(g_{x_2} \partial_{x_3} - g_{x_3} \partial_{x_2})|^2 + |(f_{x_3} \partial_{x_2} - f_{x_2} \partial_{x_3})|^2] \frac{1}{|f|^2 + |g|^2}. \tag{A.7}$$

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